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# Solving the Vertex Cover Problem with a Wave Digital Model of an Ising Machine

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**Abstract** - The efficient solution of NP-problems is an unresolved computational challenge with many real-world applications. Ising machines are promising for solving these types of problems. The idea is to map a problem onto the Ising Hamiltonian and let an Ising machine find the ground state, which corresponds to the solution of the problem. These machines are designed so they have the natural tendency to converge to the ground state of the Hamiltonian. Multidimensional wave digital algorithms are known to be massively parallel, and they are additionally robust for emulating large electrical networks, like the coupled oscillator network of an Ising machine. In this work, a wave digital model mimicking the phase dynamics of an ideal Ising machine is derived and generalized to support solving Ising problems containing the Zeeman term. To prove usefulness and quality of this wave digital Ising machine, we solve a vertex cover problem.

## Introduction

### NP-Problems

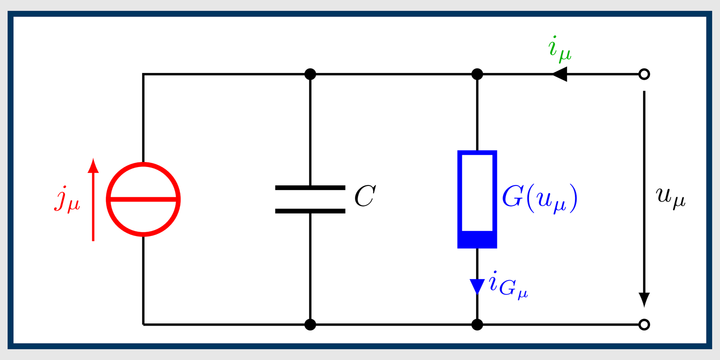
- **N**ondeterministic **P**olynomial
- Problems are not solvable in $\mathcal{O}(n^k)$, $k \in \mathbb{N}$
- Poses a challenge to the conventional van Neumann architecture
- Many NP-problems are relevant to practice

### Oscillator-based Ising Machine

- Based on the Ising model
- (Binary) Oscillators act similar to binary spins of the Ising model
- Capable of solving NP-problems
- Runtime efficient due to massive parallelism

## Circuit Synthesis

### Modified Integrator Circuit




- **S**ub**h**armonic **I**njection **L**ocking (SHIL) enforces binary states
- Modified integrator circuit mimics the phase dynamics of binary oscillator

### Phase Model

- Kuramoto model modified to account for SHIL:

$$\Sigma: \begin{cases} \dot{\varphi}_\mu = \Delta\omega_\mu + k_c\,\omega_{c_\mu} - k_s\,\omega_{s_\mu} \\ \Delta\omega_\mu = \omega_\mu - \omega_0 \\ \omega_{c_\mu} = \omega_0 \sum\limits_{\nu=1}^{n} J_{\mu\nu} \sin(\varphi_\nu - \varphi_\mu) \\ \omega_{s_\mu} = \omega_0 \sin(2\varphi_\mu) \end{cases}$$

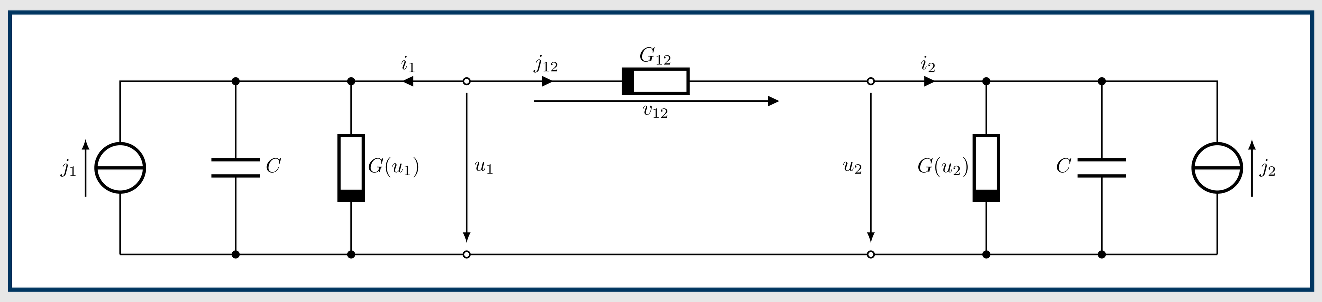
- $\Delta\omega_\mu$: frequency degeneration
- $\omega_{c_\mu}$: inter-oscillator interaction
- $\omega_{s_\mu}$: effects of SHIL
- $k_c$: coupling strength between oscillators
- $k_s$: strength of SHIL signal

### Coupling Network

- Integrators coupled according to graph topology of underlying problem
- Nonlinear coupling conductances implement nonlinear coupling of Kuramoto model
- Compact notation of coupling network ($\boldsymbol{N}$ is the incidence matrix):

$$\boldsymbol{i} = -\boldsymbol{N}\boldsymbol{j} \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{N}^\mathsf{T}\boldsymbol{u}\,;$$
$$\boldsymbol{j} = \boldsymbol{G}(\boldsymbol{v})\boldsymbol{v}\,, \quad \text{with} \quad \boldsymbol{G}(\boldsymbol{v}) = \mathrm{diag}(G_{\mu\nu})\,;$$
$$\boldsymbol{i} = \hat{\boldsymbol{G}}(\boldsymbol{v})\boldsymbol{u}\,, \quad \text{with} \quad \hat{\boldsymbol{G}}(\boldsymbol{v}) = \boldsymbol{N}\boldsymbol{G}(\boldsymbol{v})\boldsymbol{N}^\mathsf{T}$$

### Example: Two Coupled Integrators




## Wave Digital Model

### Wave Digital Concept

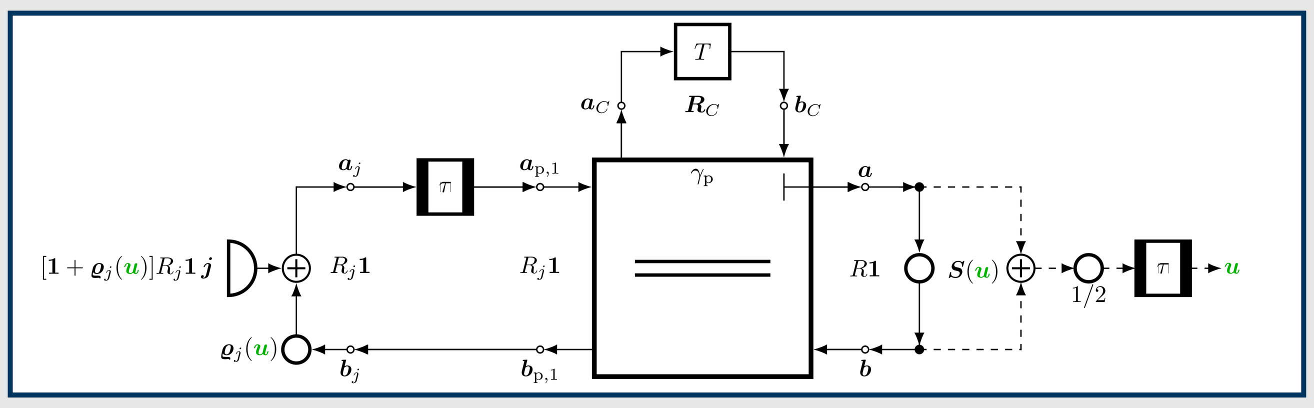
- $\boldsymbol{a} = \boldsymbol{u} + R\boldsymbol{i}$, $\quad \boldsymbol{b} = \boldsymbol{u} - R\boldsymbol{i}$, $\quad R > 0$
- Port-wise decomposition of reference circuit
- Port-wise translation into wave digital domain
- Realtime capable
- Appropriate placement of iterator elements eliminates delay-free loops

### Port-wise Translation

- Capacitor → delay
- Resistive current source → reflective source
- Parallel interconnection → parallel adaptor
- Coupling network → scattering matrix

$$\boldsymbol{S}(\boldsymbol{u}) = [\boldsymbol{G}_0 + \hat{\boldsymbol{G}}(\boldsymbol{N}^\mathsf{T}\boldsymbol{u})]^{-1} + [\boldsymbol{G}_0 - \hat{\boldsymbol{G}}(\boldsymbol{N}^\mathsf{T}\boldsymbol{u})]^{-1}\,,$$
$$\boldsymbol{G}_0 = R^{-1}\mathbf{1}$$

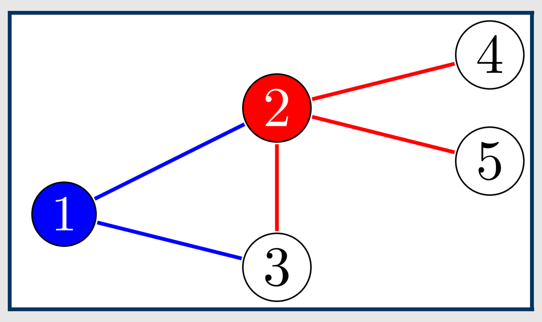
### Wave Flow Diagram of Coupled Modified Integrator Circuits



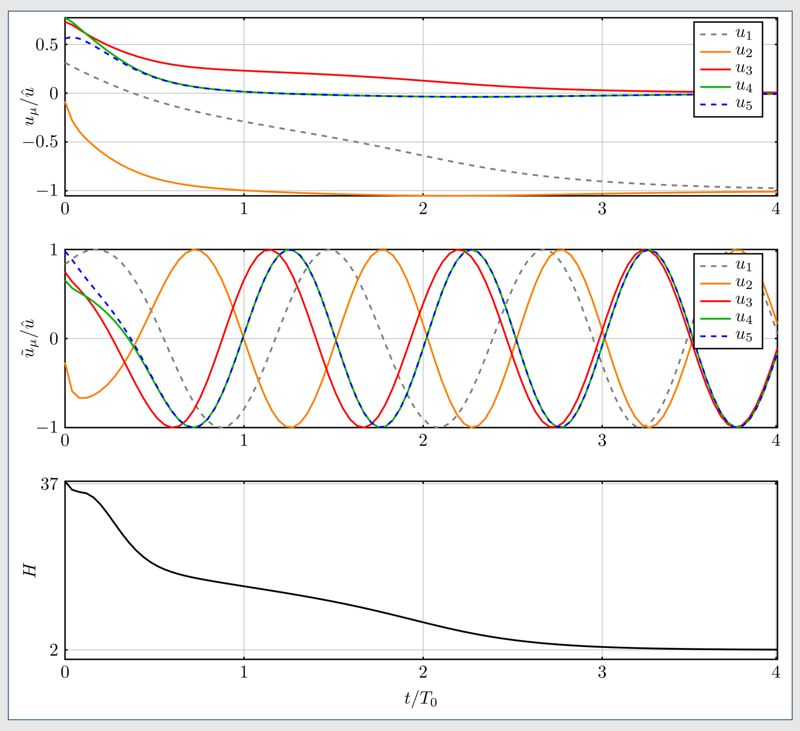

## Emulation Results

### Minimal Vertex Cover

- Find smallest vertex set covering all edges
- Two (optimal) solution sets exist of which only one is depicted
- Blue vertex covers blue edges, while red vertex covers red edges




### Ising Machine Solution




- Top: normalized voltages approach two discrete states
- Center:
  - ▶ Two oscillators synchronize to an active state (positive spin state):
    $u_1 = u_2 = -\hat{u}\,, \quad \text{with} \quad \hat{u} = 1\mathrm{V}$
  - ▶ Three remaining oscillators synchronize to an inactive state (negative spin state):
    $u_3 = u_4 = u_5 = u_6 = 0\mathrm{V}$
- Bottom: Hamiltonian energy converges to a minimum, when solution is found

**Ising machine finds a minimal vertex cover!**

## Conclusion

### Circuit Synthesis and Wave Digital Modeling

- Kuramoto model modified to account for SHIL
- Phase dynamics of binary oscillator modeled by modified integrator circuit
- Wave digital model for arbitrary network topology

### Wave Digital Ising Machine

- Can emulate the phase dynamics of an oscillator-based Ising machine in realtime
- Successfully solves the minimal vertex cover
- Displays fast convergence behavior

## Acknowledgment

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